On Electromagnetic Spinors and Electron Theory

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The relationship between the Dirac theory and electromagnetic spinors is extended to the case of finite mass. Certain products of the electromagnetic fields give rise to the Dirac differential operator upon the usual subsitutions for the energy and momentum. By placing mass in the proper place for the wave mechanical approach to quantum theory, the algebra of the fields, interpreted as quantum operators, may be deduced.

It is nearly as easy to derive the Dirac equation starting from electrodynamics as it is by starting from mechanics. The electromagnetic fields always satisfy the relation

$$(H^2+E^2)^2-(2\,\boldsymbol{E}\times\boldsymbol{H})^2 \equiv \mathscr{E}^2-P^2 = (H^2-E^2)^2+4(\boldsymbol{E}\cdot\boldsymbol{H})^2. \quad (1)$$

We may set $\boldsymbol{E}\cdot\boldsymbol{H}=0$, identify (H^2-E^2) as the mass, eventually use the substitutions $\mathscr{E}\to i\frac{\partial}{\partial t}$, $\boldsymbol{P}\to -i\nabla$, and proceed in the usual manner [1]. The only new feature here is the possibility of speculating that mass may have an electromagnetic origin. The electromagnetic approach, however, becomes considerably more interesting by introducing the concept of an electromagnetic spinor [2].

A spinor may be viewed as arising from a vector of zero length [3]. For a three-vector $\boldsymbol{x}$, we thus require

$$x_1^2+x_2^2+x_3^2=0. \quad (2)$$

Then define two numbers ξ_0, ξ_1 such that

$$x_1=\xi_0^2-\xi_1^2,\ \ x_2=i(\xi_0^2+\xi_1^2),\ \ x_3=-2\,\xi_0\,\xi_1, \quad (3)$$

which are consistent with

$$\xi_0=\pm\sqrt{\frac{x_1-i\,x_2}{2}},\quad \xi_1=\sqrt{\frac{-x_1-i\,x_2}{2}}. \quad (4)$$

The two component object $\boldsymbol{\xi}=(\xi_0,\xi_1)$ transforms as a spinor. The zero length vector and its associated spinor are also related by

$$(\boldsymbol{\sigma}\cdot\boldsymbol{x})\,\boldsymbol{\xi}=0, \quad (5)$$

which will prove useful below.

Let us define the fields

$$\boldsymbol{\Phi}_R=\boldsymbol{\sigma}\cdot(\boldsymbol{H}-i\,\boldsymbol{E}),\quad \boldsymbol{\Phi}_L=\boldsymbol{\sigma}\cdot(\boldsymbol{H}+i\,\boldsymbol{E}), \quad (6)$$

where the subscripts are suggested by the fact that $\boldsymbol{E}$ changes sign under spatial inversion, but $\boldsymbol{H}$ is invariant. Then with the identity

$$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b})=\boldsymbol{a}\cdot\boldsymbol{b}+i\,\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b}) \quad (7)$$

the requirement that the electromagnetic field have zero lenth at any point of space-time may be written as

$$\boldsymbol{\Phi}_L\boldsymbol{\Phi}_R+\boldsymbol{\Phi}_R\boldsymbol{\Phi}_L=2(H^2+E^2)=0. \quad (8)$$

The corresponding electromagnetic spinors $\boldsymbol{\varphi}_L$, $\boldsymbol{\varphi}_R$ will, in view of (5), satisfy

$$\boldsymbol{\Phi}_L\boldsymbol{\varphi}_L=0,\quad \boldsymbol{\Phi}_R\boldsymbol{\varphi}_R=0. \quad (9)$$

The individual terms in (8) are

$$\boldsymbol{\Phi}_L\boldsymbol{\Phi}_R=H^2+E^2-2\,\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{H})=\mathscr{E}-\boldsymbol{\sigma}\cdot\boldsymbol{P}, \quad (10)$$

$$\boldsymbol{\Phi}_R\boldsymbol{\Phi}_L=H^2+E^2+2\,\boldsymbol{\sigma}\cdot(\boldsymbol{E}\times\boldsymbol{H})=\mathscr{E}+\boldsymbol{\sigma}\cdot\boldsymbol{P}, \quad (11)$$

which clearly give the Dirac differential operator with the substitutions for $\mathscr{E}$, $\boldsymbol{P}$ as above. We further note that the Lagrangian density may be written as

$$\boldsymbol{\Phi}_L\boldsymbol{\Phi}_L=(H^2-E^2)+2\,i(\boldsymbol{E}\cdot\boldsymbol{H}), \quad (12)$$

$$\boldsymbol{\Phi}_R\boldsymbol{\Phi}_R=(H^2-E^2)-2\,i(\boldsymbol{E}\cdot\boldsymbol{H}), \quad (13)$$

From (10)–(13) it follows that

$$\boldsymbol{\Phi}_L\boldsymbol{\Phi}_R\boldsymbol{\Phi}_R\boldsymbol{\Phi}_L=\boldsymbol{\Phi}_R\boldsymbol{\Phi}_R\boldsymbol{\Phi}_L\boldsymbol{\Phi}_L, \quad (14)$$

which is identical to (1) (and presumably corresponds to the Klein-Gordon equation in the quantum interpretation).

Giving the wave mechanical interpretation to the right sides of (10) and (11), and following Feynman [4], applying $(\mathscr{E}-\boldsymbol{\sigma}\cdot\boldsymbol{P})$ to the right-handed spinor gives an object that transforms as a left handed spinor, and similarly for $(\mathscr{E}+\boldsymbol{\sigma}\cdot\boldsymbol{P})$. Setting

$$(\mathscr{E}-\boldsymbol{\sigma}\cdot\boldsymbol{P})\,\boldsymbol{\varphi}_R=m\,\boldsymbol{\varphi}_L, \quad (15)$$

$$(\mathscr{E}+\boldsymbol{\sigma}\cdot\boldsymbol{P})\,\boldsymbol{\varphi}_L=m\,\boldsymbol{\varphi}_R, \quad (16)$$

gives coupled equations that transform correctly, and these together are the Dirac equation in the Weyl

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representation. But from the left sides of (10) and (11) we can equally well write these as

$$(\boldsymbol{\Phi}_{\mathrm{L}}\,\boldsymbol{\Phi}_{\mathrm{R}})\,\varphi_{\mathrm{R}} = m\,\varphi_{\mathrm{L}}, \tag{17}$$

$$(\boldsymbol{\Phi}_{\mathrm{R}}\,\boldsymbol{\Phi}_{\mathrm{L}})\,\varphi_{\mathrm{L}} = m\,\varphi_{\mathrm{R}}. \tag{18}$$

Next, using (9) the left sides of these equations can only give zero. To avoid this let us introduce the interpretation familiar from quantum field theory according to which the fields (6) annihilate their corresponding spinors, giving the vacuum state:

$$\boldsymbol{\Phi}_{\mathrm{L}}\,|\varphi_{\mathrm{L}}\rangle = |0\rangle, \quad \boldsymbol{\Phi}_{\mathrm{R}}\,|\varphi_{\mathrm{R}}\rangle = |0\rangle. \tag{19}$$

Now (17) and (18) imply

$$\boldsymbol{\Phi}_{\mathrm{L}}\,|0\rangle = m\,|\varphi_{\mathrm{L}}\rangle, \quad \boldsymbol{\Phi}_{\mathrm{R}}\,|0\rangle = m\,|\varphi_{\mathrm{R}}\rangle. \tag{20}$$

Whence, $\boldsymbol{\Phi}_{\mathrm{L}}$ and $\boldsymbol{\Phi}_{\mathrm{R}}$ both annihilate and create their corresponding spinors. Note that this result is consistent with the expressions for the Lagrangian density, (12) and (13), if $\boldsymbol{E} \cdot \boldsymbol{H} = 0$, and $(H^2 - E^2)$ is taken as the mass.

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[2] W. H. Inskeep, Z. Naturforsch. **43 a,** 695 (1988).

[3] E. Cartan, The Theory of Spinors, M.I.T. Press, Cambridge 1966.

[4] R. P. Feynman, Theory of Fundamental Processes, W. A. Benjamin, New York 1962, pp. 112–115.